Scaling parameters for hypersonic flow: correlation of sphere drag data

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Abstract. Various parameters have been suggested as correlation parameters for high-speed rarefied flow, or as indicators of when rarefaction effects make the Navier-Stokes equations invalid. We consider Tsien's (1946) parameter, Cheng's (1961) rarefaction parameter, Bird's (1971) breakdown parameter, and a form of the viscous interaction parameter from shock-boundary layer theory. We show how all these parameters may be derived from the Boltzmann equation and interpreted in terms of the ratio of mean time between molecular collisions and a characteristic flow time, or as the ratio of typical shear stress to pressure in the flow. Tsien's parameter, $M_\infty/\sqrt{Re_\infty}$, is proportional to the square root of $M_\infty Kn_\infty$ (or $S_\infty Kn_\infty$) and is a better correlation parameter than the Knudsen number alone Kn_∞. Cheng's parameter, $C^* M_\infty^2/Re_\infty$, as well as the closely related viscous interaction parameter, is a modified form of Tsien's parameter. The modification factor $C^* = \mu^* T_\infty/(\mu_\infty T^*)$, accounts for the effective molecular size or collision cross-section in a characteristic region of the flow. We consider the drag coefficient for spheres in hypersonic flow as calculated with DSMC by various authors and show that Cheng's appears to be the best correlation parameter for this data.

Keywords: Kinetic Theory of Gasses, DSMC, Direct Simulation, Navier-Stokes Equations, Boltzmann Equation
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INTRODUCTION

Fig. 1 shows DSMC data for the drag coefficient of spheres calculated by various authors [5, 7, 12] for a range of Knudsen numbers, λ_∞/D, where λ_∞ is the *nominal* mean free path in the freestream, and D is the sphere diameter. The nominal mean free path was derived from the theoretical viscosity, $\mu_\infty = \mu(T_\infty)$ of the DSMC collision model, evaluated for the freestream temperature T_∞ as

$$\lambda_\infty = 2\mu_\infty/(\rho_\infty c_\infty) \tag{1}$$

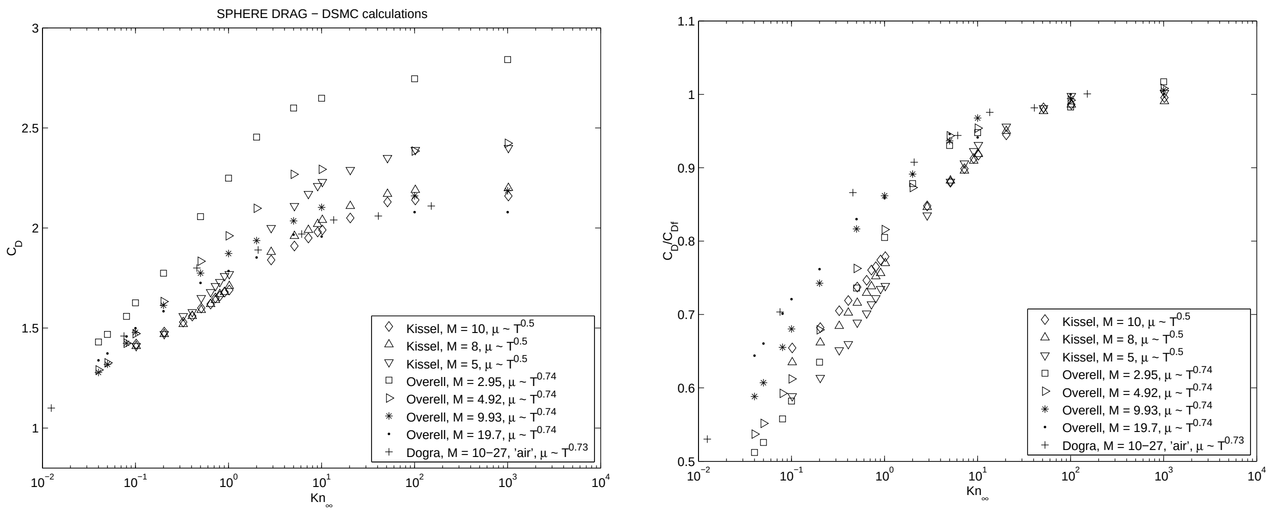


FIGURE 1. Drag coefficient C_D for cold wall spheres ($T_w \approx T_\infty$) in supersonic flow as calculated by DSMC. The data from Dogra *et al.* [5] is for a multi-species air model. The data of Kissel [7] and Overell [12] is for a pure gas with ratio of specific heats $\gamma = 5/3$.

where ρ_∞ is the freestream mass density and $c_\infty = (8RT_\infty/\pi)^{1/2}$. The 'cold-wall' condition ($T_w \approx T_\infty$) was applied in all calculations. The 'free-molecular' drag coefficient C_{Df} (*i.e.* for Kn $\to \infty$) is a function of freestream speed ratio $U_\infty/(2RT_\infty)^{1/2}$, and the wall temperature ratio T_w/T_∞, and varies from 2.2 - 2.5 for this data. As shown in the second part of Fig. 1, when the data is expressed as C_D/C_{Df} the correlation in the highly rarefied regime is improved, as it must be, but it is clear that the Knudsen number is generally not a good correlation parameter for this data. The data does not 'collapse' into a single-valued (or close to single-valued) empirical function $C_D/C_{Df} = f(Kn_\infty)$; there is strong dependence on Mach number in the results. This Mach number dependence is not surprising since, although the Knudsen number can be expressed in terms of the Mach number and Reynolds number,

$$Kn_\infty = \frac{\lambda_\infty}{L} \propto \frac{\mu_\infty}{\rho_\infty \bar{c} L} \propto \frac{U_\infty/c_\infty}{\rho_\infty U_\infty L/\mu_\infty} \propto \frac{M_\infty}{Re_\infty}, \tag{2}$$

the Knudsen number is not a flow parameter for supersonic flow in the sense that the Mach number and Reynolds numbers are flow parameters; the Knudsen number is in effect a state parameter, a measure of the gas density and molecular size (or equivalently the gas viscosity) for the given temperature. Various parameters, other than the Knudsen number, have been suggested as 'breakdown parameters' (*i.e.* as measures of when rarefaction effects make the Navier-Stokes equations invalid) or as correlation parameters for high speed rarefied flow data. We will review some of these parameters, show their near equivalence and how they are derived from the Boltzmann equation.

NON-DIMENSIONAL BOLTZMANN EQUATION

A rarefied flow can be described by the velocity distribution function f which satisfies the Boltzmann equation

$$\frac{\partial nf}{\partial t} + \mathbf{v} \cdot \frac{\partial nf}{\partial \mathbf{r}} = \Delta [nf]_{coll}$$

where t is time, $\mathbf{r}$ denotes position in space, $n = n(\mathbf{r})$ is the number density of molecules (molecules/volume), $\mathbf{v}$ is molecular velocity, $f(\mathbf{v})\,d\mathbf{v}$ is the fraction of molecules in the element of space $\mathbf{r} \to \mathbf{r} + d\mathbf{r}$ having velocity in the range $\mathbf{v} \to \mathbf{v} + d\mathbf{v}$, and $\Delta [nf]_{coll}$ is the 'collision term' denoting the rate of change of f as the result of binary intermolecular collisions at point $\mathbf{r}$. We will use the Bhatnagar-Gross-Krook (BGK) approximation and write

$$\Delta [nf]_{coll} = \frac{1}{\tau_c} (nf_e - nf)$$

where f_e is the local Maxwell distribution and τ_c is a local characteristic collision time (or relaxation time). If we now select a reference gas state (density and temperature) of (n_r, T_R), a reference collision time τ_r, a reference speed U_r and a characteristic length L_r we can write the non-dimensional BGK-Boltzmann equation as

$$\frac{\partial \hat{n}\hat{f}}{\partial \hat{t}} + \hat{\mathbf{v}} \cdot \frac{\partial \hat{n}\hat{f}}{\partial \hat{\mathbf{r}}} = \frac{1}{\zeta_r} \frac{\tau_r}{\tau_c} \left(\hat{n}\hat{f}_e - \hat{n}\hat{f} \right) \quad \text{where} \quad \zeta_r = \frac{U_r \tau_r}{L_r}. \tag{3}$$

The non-dimensional variables are $\hat{\mathbf{r}} = \mathbf{r}/L_r$, $\hat{\mathbf{v}} = \mathbf{v}/U_r$, $\hat{n} = n/n_r$, $\hat{t} = tU_r/L_r$ and $\hat{f} = fU_r^3$. For an equilibrium state ($\hat{f} \to \hat{f}_e$), the collision time can be expressed in terms of the pressure nkT and viscosity μ as $\tau = \mu(T)/(nkT)$, where k is Boltzmann's constant. As an approximation we will assume this relation holds for any $\hat{f}$, where T is the kinetic temperature. Thus, putting $\hat{F} \equiv \hat{n}\hat{f}$, Eq. 3 may be approximated as

$$\frac{\partial \hat{F}}{\partial \hat{t}} + \hat{\mathbf{v}} \cdot \frac{\partial \hat{F}}{\partial \hat{\mathbf{r}}} = \frac{1}{\zeta} \left(\frac{\mu_r}{\mu} \frac{T}{T_r} \right) \hat{n} \left(\hat{F}_e - \hat{F} \right) = \frac{1}{\zeta C} \hat{n} \left(\hat{F}_e - \hat{F} \right) \quad \text{where} \quad C = \frac{\mu T_r}{\mu_r T}. \tag{4}$$

A natural choice of reference state for the supersonic flow round a sphere is the freestream flow, and the natural choice of L_r is the sphere diameter D. Then

$$\frac{\partial \hat{F}}{\partial \hat{t}} + \hat{\mathbf{v}} \cdot \frac{\partial \hat{F}}{\partial \hat{\mathbf{r}}} = \frac{1}{\zeta_\infty C} \hat{n} \left(\hat{F}_e - \hat{F} \right) \text{ where } \zeta_\infty = \frac{U_\infty \tau_\infty}{D} \text{ and } C = \frac{\mu T_\infty}{\mu_\infty T}. \tag{5}$$

The parameter $\zeta_\infty = U_\infty \tau_\infty / D = \Lambda_\infty / D$ might be called the 'body-relative Knudsen number', since $\Lambda_\infty = U_\infty \tau_\infty$ is the average distance moved, relative to the body, by a freestream molecule between collisions, as illustrated in Fig. 2. Note

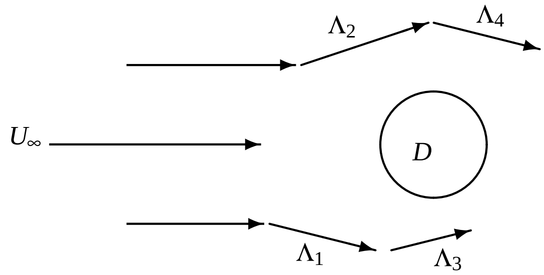


FIGURE 2. Typical paths of molecules as they move past the body. The distance the molecules move between collision, relative to the body, is denoted by Λ_j. The mean free path, in the body frame of reference, is $\Lambda = \sum \Lambda_j / N$. The ratio $D/\Lambda_\infty = (D/U_\infty)/\tau_\infty$ is a measure of the number of collisions suffered by a typical molecule as it moves past the body.

that $\Lambda_\infty/\lambda_\infty = U_\infty \tau_\infty/(c_\infty \tau_\infty) = U_\infty/c_\infty \approx S_\infty \approx M_\infty$ so that

$$\zeta_\infty = \frac{\Lambda_\infty}{D} \approx S_\infty Kn_\infty \approx M_\infty Kn_\infty \approx \frac{M_\infty^2}{Re_\infty}, \tag{6}$$

where Eq. 2 has been used. The parameter $M_\infty^2/Re_\infty = \zeta_\infty$ was once commonly used as a correlation parameter for rarefied flow, *e.g.* [4, 6, 13], and the parameter $M_\infty/\sqrt{Re_\infty} = \sqrt{\zeta_\infty}$ has been used as a correlation parameter for the skin friction coefficient on a flat plate in hypersonic flow [8], and for the lift-to-drag ratio for various hypersonic vehicles [14]. The latter form of this parameter was suggested by Tsien's 1946 investigation of high speed rarefied flow, which he referred to as 'superaerodynamics' [15].

Tsien's parameter

Consider Eq. 5 in the special case of a linear viscosity law, $\mu = \mu_\infty (T/T_\infty)$, so that $C = 1$ for all temperatures. Then the non-dimensional Boltzmann equation is

$$\frac{\partial \hat{F}}{\partial \hat{t}} + \hat{\mathbf{v}} \cdot \frac{\partial \hat{F}}{\partial \hat{\mathbf{r}}} = \zeta_\infty^{-1} \hat{n} \left(\hat{F}_e - \hat{F} \right).$$

When the diffusely-reflecting condition is taken into account at the body surface, the non-dimensional solution of the Boltzmann equation depends on T_w/T_∞, M_∞, and Kn_∞ [10], and not just on the combination $\zeta_\infty \approx M_\infty Kn_\infty$. Nevertheless, from the form of the non-dimensional Boltzmann equation above, it is reasonable to expect that the flow might be a strong function of ζ_∞. Tsien [15] reached this conclusion by another route for hypersonic slender body flow; he showed that the Burnett equations were appreciably different from the Navier-Stokes equations when λ_∞/δ was large, where δ is the boundary layer thickness. Since for laminar flow $\delta \propto L/Re_\infty^{1/2}$, where L is the body length, Tsien's Knudsen number (or Tsien's parameter) may be written

$$Kn_\delta \equiv \frac{\lambda_\infty}{\delta} = \frac{\lambda_\infty}{L} \frac{L}{\delta} \propto Kn_\infty Re_\infty^{1/2} \propto \frac{M_\infty}{Re_\infty^{1/2}} \tag{7}$$

where Eq. 2 has been used; thus Tsien's parameter $M_\infty/\sqrt{Re_\infty}$ is just the square root of the flow parameter ζ_∞.

Fig. 3 shows Tsien's parameter used as a correlation parameter for the sphere drag data. Compared to the correlation *via* the Knudsen number, we see an improved 'collapse' of the data in the slip flow regime, which Tsien identified as $0.01 \leq M_\infty/Re_\infty^{1/2} < 1$, but little improvement for more rarefied flow ($M_\infty/Re_\infty^{1/2} > 1$).

Bird's breakdown parameter

Bird [2] showed, for a high speed expanding flow, that the onset of non-equilibrium conditions (the appearance of different kinetic temperatures for different molecular energy modes) was governed by the value of the local 'breakdown parameter' P. For unsteady flow $P = \tau |D \ln(\rho/\rho_\infty)/Dt|$. For steady flow

$$P \propto \tau U \left| \frac{\nabla \rho}{\rho} \right| = \frac{\Lambda}{L_\rho} \propto \frac{U}{\bar{c}} \frac{\lambda}{L_\rho} \propto S Kn_\rho$$

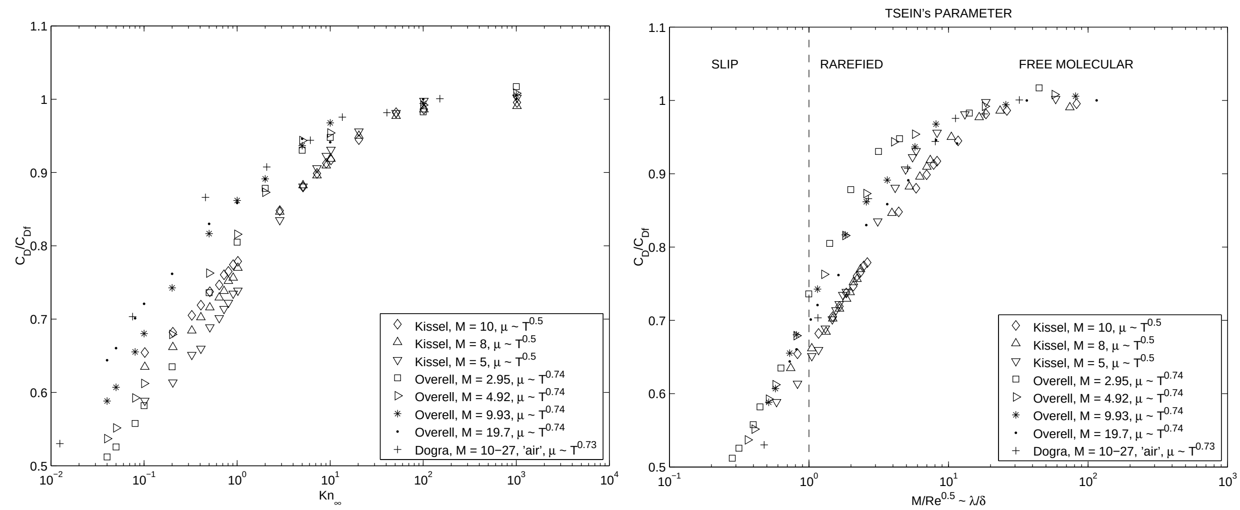


FIGURE 3. Sphere drag *vs.* Knudsen number (left) and Tsien's rarefaction parameter $M_\infty/\sqrt{Re_\infty}$. This parameter correlates the data in the in Tsien's 'slip regime' $M_\infty/\sqrt{Re_\infty} < 1$ better than the Knudsen number

where U is the flow speed, $L_\rho = |\rho/\nabla\rho|$ is a flow length derived from the density gradients, $\Lambda = U\tau$ is a local mean free path relative to the (fixed) flow gradient, S is the local speed ratio and $Kn_\rho = \lambda/L_\rho$ is the local 'gradient-length-Knudsen-number'. The local flow speed will be proportional to the freestream speed, and that the gradient flow length L_ρ will be proportional to the dimension of the body in the flow, so $P \propto S_\infty Kn_\infty \propto M_\infty^2/Re_\infty$. In other words the breakdown parameter is another form of the rarefaction parameter ζ_∞, or Tsien's parameter $M_\infty/\sqrt{Re_\infty}$. We can replace L_ρ with $L_U = |U/\nabla U|$ and τ by μ/p to $P \propto \mu\,|\nabla U|\,/p$ indicating that non-equilibrium flow is expected when the shear stress ($\propto \mu\nabla U$) is appreaciable relative to the pressure.

Cheng's parameter

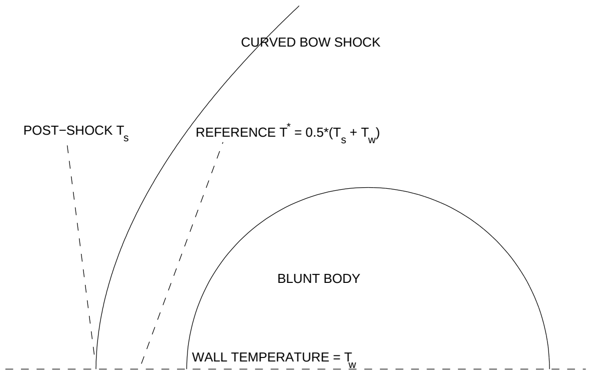
Cheng [3] considered the hypersonic flow about a blunt body and introduced the parameter

$$K_C^2 = \frac{p_\infty D}{C^* U_\infty \mu_\infty} \text{ where } C^* = \mu^* T_\infty/\mu_\infty T^*. \tag{8}$$

Here $T^* = (T_s + T_w)/2$ is a temperature characteristic of the merged shock and boundary layer ahead of the blunt body, $\mu^* = \mu(T^*)$ and D is the body dimension. As shown in Fig. 4, T_s is the temperature immediately behind the normal bow shock, and T_w is the body wall temperature. The inverse of K_C^2 is proportional to $C^*\zeta_\infty$. The Boltzmann equation in the form of Eq. 4 shows that the flow depends not just on the parameter ζ_∞, but on the local value $C\zeta_\infty$ where $C = \mu(T)/\mu(T_\infty) \times T_\infty/T$ is proportional to the varying collision rate throughout the flow. Thus we can interpret Cheng's $C^*\zeta_\infty$ as an average value or characteristic value of the term $C\zeta_\infty$ in the Boltzmann equation evaluated for a characteristic region of the flow, *i.e.* a region where the molecular collisions are assumed to have the greatest effect on the overall flow. Cheng's parameter has been used to correlate the heat transfer to blunt bodies for experimental results [11] and for DSMC results [9]. The DSMC results are shown in the first part of Fig. 5.

Various non-dimensional groups or parameters have been given the name 'viscous interaction parameter' for hypersonic slender body flow. One of these is $\overline{V}' = \sqrt{C^*}M_\infty/\sqrt{Re_\infty} \approx \sqrt{C^*\zeta_\infty}$, where T^* is a characteristic temperature in the boundary layer [1]. Thus $\overline{V}'$ is a form of Cheng's parameter. It has has been used to correlate the flight and wind-tunnel data for the axial force coefficient on the NASA shuttle [16]. A particular value of T^*, discovered by trial-and-error, gave the best correlation, which is shown in Fig. 5.

Fig. 6 shows the sphere drag data correlated with the square of Tsien's parameter $M_\infty^2/Re_\infty = \zeta_\infty$ and the inverse Cheng's parameter $C^*M_\infty^2/Re_\infty = C^*\zeta_\infty$. The correlation is improved by the factor C^*, which adjusts for the different collision frequency at the characteristic temperature T^*, compared to the freestream collision frequency.



Cheng's parameter

$$K_C^2 = \frac{p_\infty D}{C^* U_\infty \mu_\infty} = \frac{\rho_\infty R T_\infty U_\infty D}{U_\infty^2 \mu_\infty} \sim \frac{Re_\infty}{C^* M_\infty^2}$$

is a modified Tsien parameter

$$\sqrt{\frac{1}{K_C^2}} \sim \sqrt{C^*}\frac{M_\infty}{\sqrt{Re_\infty}} = \sqrt{C^*} Kn_\delta \sim \sqrt{C^* \zeta_\infty}.$$

$C^* \propto \mu(T^*)/(n^* k T^*)$, characterizes the collision time in the merged shock and boundary layer.

FIGURE 4. Characteristic temperature, merged layer: $T^* = \frac{1}{2}(T_s + T_w)$. Collision time in merged layer is $\approx \mu^*/(n^* k T^*)$

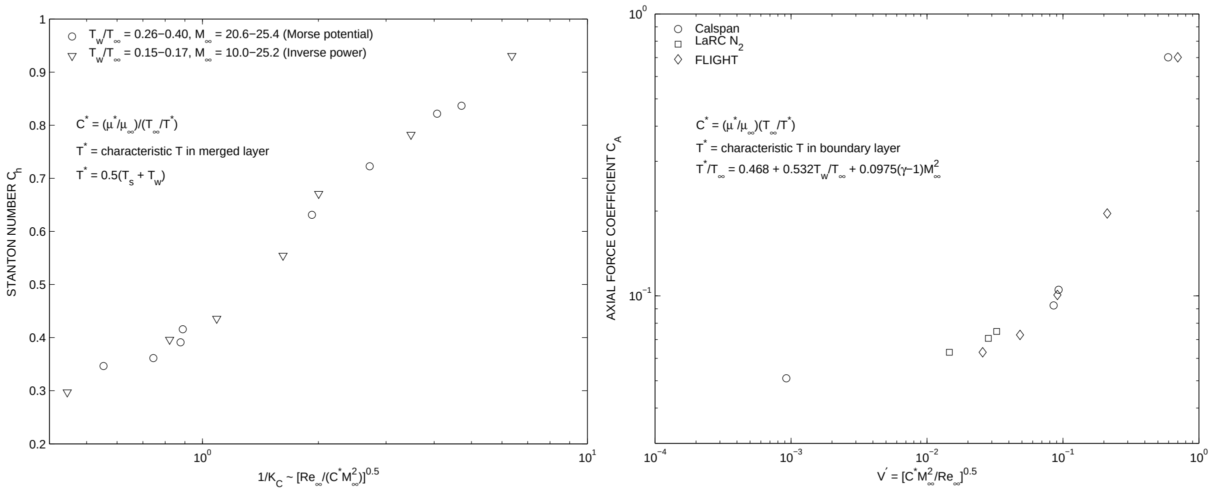


FIGURE 5. Left: Heat transfer to blunt cylinder (DSMC results [9]) *vs.* inverse Cheng's parameter. Right: Shuttle axial force coefficient (wind tunnel and flight) *vs.* modified viscous interaction parameter [16]. Parameters differ in choice of T^* only.

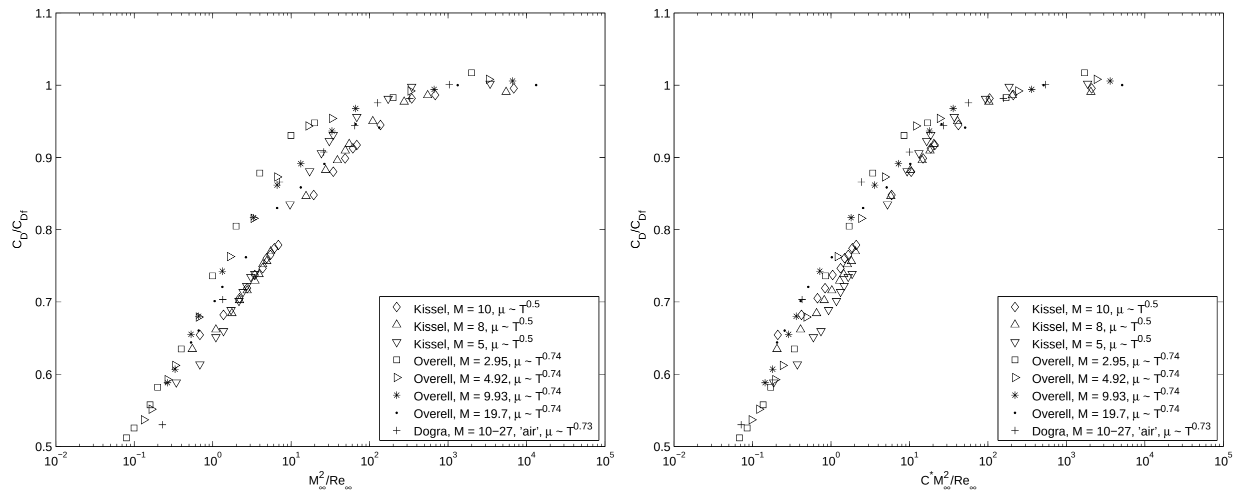


FIGURE 6. Square of Tsien's parameter (left) and inverse Cheng's parameter (right) used as a correlation parameter.

CONCLUSIONS

Various independently proposed rarefaction parameters are closely related to the non-dimensional group

$$C\zeta_\infty \approx CM_\infty^2/Re_\infty, \text{ where } C \equiv \frac{\mu/T}{\mu_\infty/T_\infty},$$

which appears on the RHS of the non-dimensional Boltzmann equation. Tsien (1946) proposed the parameter $M_\infty/\sqrt{Re_\infty} \propto \lambda_\infty/\delta$ for slender body flow, where δ is the boundary layer thickness, as a breakdown parameter. The square of Tsien's parameter M_∞^2/Re_∞ was once popular as a correlation parameter for rarefied flow. It is closely related to Bird's (1971) 'breakdown parameter', and is proportional to the expected ratio of shear stress to pressure in the flow. Cheng (1961) proposed a parameter equivalent to $C^* M_\infty^2/Re_\infty$ which we have referred to as Cheng's parameter or a modified Tsien parameter; it is similar to a viscous interaction parameter. The modification factor $C^* = \mu^* T_\infty/(\mu_\infty T^*)$ accounts for the effective molecular size at a characteristic temperature of the flow T^*. For a given flow configuration, one must select (guess) the characteristic T^* of the flow. Wilhite *et al.* [16] have given an expression for T^* for the boundary layer temperature in a special case, and Cheng gave a simple expression for T^* for blunt body flow, which was an estimate of the temperature in the merged shock/boundary layer region ahead of the blunt body. We tested the various parameters as correlation parameters for DSMC derived values for hypersonic sphere drag coefficients; Cheng's parameter is a better correlation parameter than Tsien's parameter, which in turn is better than the Knudsen number alone.

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